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Cusps of cosmic strings emit strong beams of high-frequency gravitational waves (GW). As a consequence of these beams, the stochastic ensemble of gravitational waves generated by a cosmological network of oscillating loops is strongly non Gaussian, and includes occasional sharp bursts that stand above the rms GW background. These bursts might be detectable by the planned GW detectors LIGO/VIRGO and LISA for string tensions as small as $G\mu \sim 10^{-13}$. The GW bursts discussed here might be accompanied by Gamma Ray Bursts.

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Cosmic strings are linear topological defects that could be formed at a symmetry breaking phase transition in the early universe. Strings are predicted in a wide class of elementary particle models and can give rise to a variety of astrophysical phenomena [1]. In particular, oscillating loops of string can generate a potentially observable gravitational wave (GW) background ranging over many decades in frequency. The spectrum of this stochastic background has been extensively discussed in the literature [2–7], but until now it has been tacitly assumed that the GW background is nearly Gaussian. In this paper, we show that the GW background from strings is strongly non-Gaussian and includes sharp GW bursts (GWB) emanating from cosmic string cusps [8]. We shall estimate the amplitude, frequency spectrum and rate of the bursts, and discuss their detectability by the planned GW detectors LIGO/VIRGO and LISA.

We begin with a brief summary of the relevant string properties and evolution [1]. A horizon-size volume at any cosmic time t contains a few long strings stretching across the volume and a large number of small closed loops. The typical length and number density of loops formed at time t are approximately given by

$$l \sim \alpha t, \qquad n_l(t) \sim \alpha^{-1} t^{-3}. \tag{1}$$

The exact value of the parameter α in (1) is not known. We shall assume, following [5], that α is determined by the gravitational backreaction, so that $\alpha \sim \Gamma G\mu$, where $\Gamma \sim 50$ is a numerical coefficient, G is Newton's constant, and μ is the string tension, i.e. the mass per unit length of the string. The coefficient Γ enters the total rate of energy loss by gravitational radiation $d\mathcal{E}/dt \sim \Gamma G\mu^2$. For a loop of invariant length l [9], the oscillation period is $T_l = l/2$ and the lifetime is $\tau_l \sim l/\Gamma G\mu \sim t$.

A substantial part of the radiated energy is emitted from near-cusp regions where, for a short period of time, the string reaches a speed very close to the speed of light [4]. Cusps tend to be formed a few times during each oscillation period [10]. Let us estimate the (trace-reversed) metric perturbation $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$ emitted near a cusp. Let $\kappa_{\mu\nu} = r_{\rm phys} \overline{h}_{\mu\nu}$ denote the product of the metric perturbation by the distance away from the string, estimated in the local wave-zone of the oscillating string. $\kappa_{\mu\nu}$ is given by a Fourier series whose coefficients are proportional to the Fourier transform of the stress-energy tensor of the string:

$$T^{\mu\nu}(k^\lambda) = T_l^{-1} \int_{T_l} d\tau d\sigma \mu (\dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu) \exp(-ik.X). \tag{2}$$

Here, $X^\mu(\tau,\sigma)$ represents the string worldsheet, parametrized by the conformal coordinates τ and σ. [$\dot{X} = \partial_\tau X$, $X' = \partial_\sigma X$.] In the direction of emission $\mathbf{n}$, $k^\mu = (\omega, \mathbf{k})$ runs over the discrete set of values $4\pi l^{-1} m(1,\mathbf{n})$, where $m = 1, 2, \ldots$. Near a cusp (and only near a cusp) the Fourier series giving $\kappa_{\mu\nu}$ is dominated by large m values, and can be approximated by a continuous Fourier integral. The continuous Fourier component (corresponding to an octave of frequency around f) $\kappa(f) \equiv |f| \widetilde{\kappa}(f) \equiv |f| \int dt \exp(2\pi i f t) \kappa(t)$ is then given by

$$\kappa_{\mu\nu}(f) = 2Gl|f| T_{\mu\nu}(k^\lambda). \tag{3}$$

In a conformal gauge, the string motion is given by $X^\mu = \frac{1}{2}(X_+^\mu(\sigma_+) + X_-^\mu(\sigma_-))$,where $\sigma_\pm = \tau \pm \sigma$, and where $\dot{X}_\pm^\mu$ is a null 4-vector. Further choosing a "time gauge" ($\tau = X^0 = t$) ensures that the time components of these null vectors are equal to 1. A cusp is a point on the worldsheet where these two null vectors coincide, say $\dot{X}_{c\pm}^\mu = l^\mu = (1, \mathbf{n_c})$. One can estimate the waveform (3) emitted near the cusp (localized, say, at $\sigma_+ = \sigma_- = 0$) by replacing in the integral (2) $X_\pm^\mu$ by their local Taylor expansions

$$X_\pm^\mu(\sigma_\pm) = X_{c\pm}^\mu + l^\mu \sigma_\pm + \frac{1}{2} \ddot{X}_\pm^\mu \sigma_\pm^2 + \frac{1}{6} \dddot{X}_\pm^\mu \sigma_\pm^3 + \ldots \tag{4}$$

For a given frequency $f \gg T_l^{-1}$, the integral (2) is significant only if the angle θ between the direction of emission $\mathbf{n}$ and the "3-velocity" of the cusp $\mathbf{n_c}$) is smaller than about $\theta_m \equiv (T_l|f|)^{-1/3}$. When $\theta \ll \theta_m$, the integral can be explicitly evaluated. After a suitable gauge transformation one finds [11]

$$\kappa^{\mu\nu}(f) = -CG\mu(2\pi|f|)^{-1/3}A_+^{(\mu}A_-^{\nu)}, \qquad (5)$$

where $C = 4\pi(12)^{4/3}(3\Gamma(1/3))^{-2}$ and where the linear polarization tensor is the symmetric tensor product of $A_\pm^\mu \equiv \ddot{X}_\pm^\mu/|\ddot{X}_\pm|^{4/3}$. The inverse Fourier transform of Eq.(5) gives a time-domain waveform proportional to $|t-t_c|^{1/3}$, where t_c corresponds to the peak of the burst [12]. The sharp spike at $t = t_c$ exists only if $\theta = 0$ (i.e. if one observes it exactly in the direction defined by the cusp velocity). When $0 \neq \theta \ll 1$ the spike is smoothed over $|t - t_c| \sim \theta^3 T_l$. In the Fourier domain this smoothing corresponds to an exponential decay for frequencies $|f| \gg 1/(\theta^3 T_l)$.

Eq.(5) gives the waveform in the local wave-zone of the oscillating loop: $\overline{h}_{\mu\nu} = \kappa_{\mu\nu}/r_{\rm phys}$. To take into account the subsequent propagation of this wave over cosmological distances, until it reaches us, one must introduce three modifications in this waveform: (i) replace $r_{\rm phys}$ by $a_0 r$ where r is the comoving radial coordinate in a Friedman universe (taken to be flat: $ds^2 = -dt^2 + a(t)^2(dr^2 + r^2 d\Omega^2)$) and $a_0 = a(t_0)$ the present scale factor , (ii) express the locally emitted frequency in terms of the observed one $f_{\rm em} = (1 + z)f_{\rm obs}$ where z is the redshift of the source, and (iii) transport the polarization tensor of the wave by parallel propagation (pp) along the null geodesic followed by the GW:

$$\overline{h}_{\mu\nu}(f) = \kappa_{\mu\nu}^{\rm pp}((1 + z)f)/(a_0 r). \qquad (6)$$

Here, and henceforth, $f > 0$ denotes the observed frequency. In terms of the redshift we have $a_0 r = 3t_0(1 - (1 + z)^{-1/2})$, where t_0 is the present age of the universe (this relation holds during the matter era, and can be used for the present purpose in the radiation era because $a_0 r$ has a finite limit for large z).

For our order-of-magnitude estimates we shall assume that $|\ddot{X}_\pm| \sim 2\pi/l$. The various numerical factors in the equations above nearly compensate each other to give the following simple estimate for the observed waveform in the frequency domain ($h(f) \equiv |f|\widetilde{h}(f) \equiv |f| \int dt \exp(2\pi i f t)h(t)$)

$$h(f) \sim \frac{G\mu l}{((1 + z)fl)^{1/3}} \frac{1 + z}{t_0 z}. \qquad (7)$$

Here the explicit redshift dependence is a convenient simplification of the exact one given above. This result holds only if, for a given observed frequency f, the angle $\theta = \cos^{-1}(\mathbf{n} \cdot \mathbf{n}_c)$ satisfies

$$\theta \lesssim \theta_m \equiv ((1 + z)fl/2)^{-1/3}. \qquad (8)$$

To know the full dependence of $h(f)$ on the redshift we need to express $l \sim \alpha t$ in terms of z. We write

$$l \sim \alpha t_0 \varphi_l(z); \quad \varphi_l(z) = (1 + z)^{-3/2}(1 + z/z_{eq})^{-1/2}. \quad (9)$$

Here $z_{eq} \simeq 2.4 \times 10^4 \Omega_0 h_0^2 \simeq 10^{3.9}$ is the redshift of equal matter and radiation densities, and we found it convenient to define the function $\varphi_l(z)$ which interpolates between the different functional z-dependences of l in the matter era, and the radiation era. [We shall systematically introduce such interpolating functions of z, valid for all redshifts, in the following.] Inserting Eq.(9) into Eq.(7) yields

$$h(f, z) \sim G\mu\alpha^{2/3}(ft_0)^{-1/3}\varphi_h(z),$$
$$\varphi_h(z) = z^{-1}(1 + z)^{-1/3}(1 + z/z_{eq})^{-1/3}. \qquad (10)$$

We can estimate the rate of GWBs originating at cusps in the redshift interval dz, and observed around the frequency f, as $d\dot{N} \sim \frac{1}{4}\theta_m^2(1 + z)^{-1}\nu(z)dV(z)$. Here, the first factor is the beaming fraction within the cone of maximal angle $\theta_m(f, z)$, Eq.(8), the second factor comes from the relation $dt_{\rm obs} = (1 + z)dt$, $\nu(t) \sim f_c n_l(t)/T_l \sim 2\alpha^{-2}t^{-4}$ is the number of cusp events per unit spacetime volume, $f_c \sim 1$ is the average number of cusps per oscillation period of a loop, $T_l \sim \alpha t/2$, and $dV(z)$ is the proper volume between redshifts z and $z + dz$. In the matter era $dV = 54\pi t_0^3[(1 + z)^{1/2} - 1]^2(1 + z)^{-11/2}dz$, while in the radiation era $dV = 72\pi t_0^3(1 + z_{eq})^{1/2}(1 + z)^{-5}dz$. The function $\dot{N}(f, z) \equiv d\dot{N}/d\ln z$ can be approximately represented by the following interpolating function of z

$$\dot{N}(f, z) \sim 10^2 t_0^{-1}\alpha^{-8/3}(ft_0)^{-2/3}\varphi_n(z),$$
$$\varphi_n(z) = z^3(1 + z)^{-7/6}(1 + z/z_{eq})^{11/6}. \qquad (11)$$

The observationally most relevant question is: what is the typical amplitude of bursts $h_{\dot{N}}^{\rm burst}(f)$that we can expect to detect at some given rate $\dot{N}$, say, one per year? Using $\dot{N} = \int_0^{z_m} \dot{N}(f, z)d\ln z \sim \dot{N}(f, z_m)$, where z_m is the largest redshift contributing to $\dot{N}$, one can estimate $h_{\dot{N}}^{\rm burst}(f)$ by solving for z in Eq.(11) and substituting the result $z = z_m(\dot{N}, f)$ in Eq.(10). The final answer has a different functional form depending on the magnitude of the quantity

$$y(\dot{N}, f) \equiv 10^{-2}\dot{N}t_0\alpha^{8/3}(ft_0)^{2/3}. \qquad (12)$$

Indeed, if $y < 1$ the dominant redshift will be $z_m(y) < 1$; while, if $1 < y < z_{eq}{}^{11/6}$, $1 < z_m(y) < z_{eq}$, and if $y > z_{eq}{}^{11/6}$, $z_m(y) > z_{eq}$. We can again introduce a suitable interpolating function $g(y)$ to represent the final result as an explicit function of y:

$$h_{\dot{N}}^{\rm burst}(f) \sim G\mu\alpha^{2/3}(ft_0)^{-1/3}g[y(\dot{N}, f)],$$
$$g(y) = y^{-1/3}(1 + y)^{-13/33}(1 + y/(z_{eq})^{11/6})^{3/11}. \qquad (13)$$

The prediction Eq.(13) for the amplitude of the GWBs generated at cusps of cosmic strings is the main new result of this work. To see whether or not these bursts can be distinguished from the stochastic gravitational wave background, we have to compare the burst amplitude (13) to the rms amplitude of the background, $h_{\rm rms}$, at the same frequency. We define $h_{\rm rms}$ as the "confusion" part of the ensemble of bursts Eq.(13), i.e. the superposition of all the "overlapping" bursts, those whose occurrence rate is higher than their typical frequency. This is estimated by multiplying the square of Eq.(10) by the number of overlapping bursts within a frequency octave $n_z(f) \equiv f^{-1}\dot{N}(f,z)$, and then integrating over all $\ln z$ such that $n_z(f) > 1$, and $\theta_m(f,z) < 1$:

$$h^2_{\rm rms}(f) = \int h^2(f,z) n_z(f) d\ln z H(n_z - 1) H(1-\theta_m), \tag{14}$$

where H denotes Heaviside's step function. Eq.(14) differs from previous estimates of the stochastic background [2–7] (beyond the fact that we use the simplified loop density model Eq.(1)) in that the latter did not incorporate the restriction to $n_z(f) > 1$, i.e. they included non-overlapping bursts in the average of the squared GW amplitude.

It is easily checked from Eq.(13) that $h^{\rm burst}$ is a monotonically decreasing function of both $\dot{N}$ and f. These decays can be described by (approximate) power laws , with an index which depends on the relevant range of dominant redshifts; e.g., as $\dot{N}$ increases, $h^{\rm burst}$ decreases first like $\dot{N}^{-1/3}$ (in the range $z_m < 1$), then like $\dot{N}^{-8/11}$ (when $1 < z_m < z_{eq}$), and finally like $\dot{N}^{-5/11}$ (when $z_m > z_{eq}$). For the frequency dependence of $h^{\rm burst}$, the corresponding power-law indices are successively: $-5/9, -9/11$ and $-7/11$. [These slopes come from combining the basic $f^{-1/3}$ dependence of the spectrum of each burst with the indirect dependence on f of the dominant redshift $z_m(\alpha, \dot{N}, f)$.] By contrast, when using our assumed link $G\mu \sim \alpha/50$ between the string tension μ and the parameter α, one finds that the index of the power-law dependence of $h^{\rm burst}$ upon α takes successively the values $+7/9, -3/11$ and $+5/11$. Therefore, in a certain range of values of α (corresponding to $1 < z_m(\alpha, \dot{N}, f) < z_{eq}$) the GWB amplitude (paradoxically) *increases* as one decreases α , i.e. $G\mu$.

In Fig. 1 we plot (as a solid line) the logarithm of the GW burst amplitude, $\log_{10}(h^{\rm burst})$, as a function of $\log_{10}(\alpha)$, for $\dot{N} = 1$ yr^{-1}, and for $f = f_c = 150$ Hz. This central frequency is the optimal one for the detection of a $f^{-1/3}$-spectrum burst by LIGO. We indicate on the same plot (as horizontal dashed lines) the (one sigma) noise levels $h^{\rm noise}$ of LIGO 1 (the initial detector), and LIGO 2 (its planned advanced configuration). The VIRGO detector has essentially the same noise level as LIGO 1 for the GW bursts considered here. These noise levels are defined so that the integrated optimal (with a matched filter $\propto |f|^{-1/3}$) signal to noise ratio (SNR) for each detector is $SNR = h^{\rm burst}(f_c)/h^{\rm noise}$. The short-dashed line in the lower right corner is the rms GW amplitude, Eq.(14). One sees that the burst amplitude stands well above the stochastic background [14]. Clearly the search by LIGO/VIRGO of the type of GW bursts discussed here is a sensitive probe of the existence of cosmic strings in a larger range of values of α than the usually considered search for a stochastic GW background.

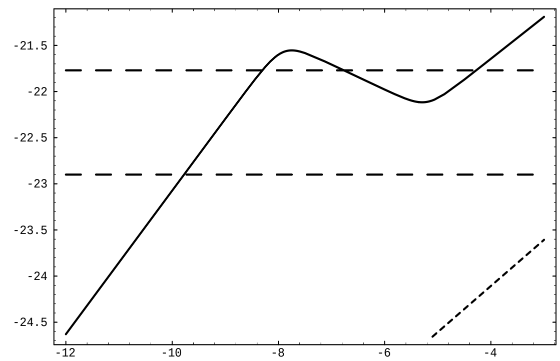

FIG. 1. Gravitational wave amplitude of bursts emitted by cosmic string cusps in the LIGO/VIRGO frequency band, as a function of the parameter $\alpha = 50G\mu$. [In a base-10 log-log plot.] The horizontal dashed lines indicate the one sigma noise levels (after optimal filtering) of LIGO 1 (initial detector) and LIGO 2 (advanced configuration).The short-dashed line indicates the rms amplitude of the stochastic GW background.

From Fig. 1 we see that the discovery potential of ground-based GW interferometric detectors is richer than hitherto envisaged, as it could detect cosmic strings in the range $\alpha \gtrsim 10^{-10}$, i.e. $G\mu \gtrsim 10^{-12}$. Let us also note that the value of α suggested by the (superconducting-) cosmic-string Gamma Ray Burst (GRB) model of Ref. [13], namely $\alpha \sim 10^{-8}$, nearly corresponds, in Fig. 1, to a local maximum of the GW burst amplitude. [This local maximum corresponds to $z_m \sim 1$. The local minimum on its right corresponds to $z_m \sim z_{eq}$.] In view of the crudeness of our estimates, it is quite possible that LIGO 1/VIRGO might be sensitive enough to detect these GW bursts. Indeed, if one searches for GW bursts which are (nearly) coincident with (some [15]) GRB the needed threshold for a convincing coincident detection is much closer to unity than in a blind search. [In a blind search, by two detectors, one probably needs SNRs ~ 4.4 to allow for the many possible arrival times. Note that the optimal filter, $h^{\rm template}(f) = |f|^{-1/3}$, for our GWBs is parameter-free.]

In Fig. 2 we plot $\log_{10}(h^{\rm burst})$ as a function of $\log_{10}(\alpha)$ for $\dot{N} = 1$ yr^{-1}, and for $f = f_c = 3.9 \times 10^{-3}$ Hz. This frequency is the optimal one for the detection of a $f^{-1/3}$ GWB by the planned spaceborne GW detector LISA. [In determining the optimal SNR in LISA we combined the latest estimate of the instrumental noise [16] with estimates of the galactic confusion noise [17].] Fig. 2 compares $h^{\rm burst}(f_c)$ to both LISA's (filtered) noise level $h^{\rm noise}$ and to the cosmic-string-generated stochastic background $h^{\rm rms}$, Eq.(14). The main differences with the previous plot are: (i) the signal strength, and the SNR, are typically much higher for LISA than for LIGO, (ii) though the GW burst signal still stands out well above the rms background, the latter is now higher than the (broad-band) detector noise in a wide range of values of α. LISA is clearly a very sensitive probe of cosmic strings. It might detect GWBs for values of α as small as $\sim 10^{-11.6}$. [Again, a search in coincidence with GRBs would ease detection. Note, however, that, thanks to the lower frequency range, even a blind search by the (roughly) two independent arms of LISA would need a lower threshold, ~ 3, than LIGO.]

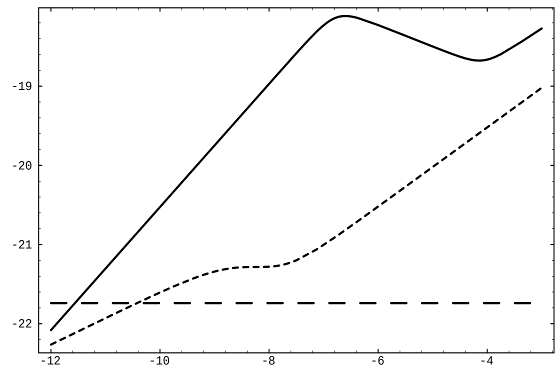

FIG. 2. Gravitational wave amplitude of bursts emitted by cosmic string cusps in the LISA frequency band, as a function of the parameter $\alpha = 50G\mu$. [In a base-10 log-log plot.] The short-dashed curve indicates the rms amplitude of the stochastic GW background. The lower long-dashed line indicates the one sigma noise level (after optimal filtering) of LISA.

We shall discuss elsewhere the consequences for the interpretation of the pulsar timing experiments of the GWB-induced non-Gaussianity of the stochastic GW background [11].

When comparing our results with observations, one should keep in mind that the model we used for cosmic strings involves a number of simplifying assumptions. *(i)* All loops at time t were assumed to have length $l \sim \alpha t$ with $\alpha \sim \Gamma G\mu$. It is possible, however, that the loops have a broad length distribution $n(l,t)$ and that the parameter α characterizing the typical loop length is in the range $\Gamma G\mu < \alpha \lesssim 10^{-3}$. (Here, the upper bound comes from numerical simulations and the lower bound is due to the gravitational backreaction.) *(ii)* We also assumed that the loops are characterized by a single length scale l, with no wiggliness on smaller scales. Short-wavelength wiggles on scales $\ll \Gamma G\mu t$ are damped by gravitational backreaction, but some residual wiggliness may survive. As a result, the amplitude and the angular distribution of gravitational radiation from cusps may be modified. *(iii)* We assumed the simple, uniform estimate Eq.(1) for the space density of loops. This estimate may be accurate in the matter era but is probably too small by a factor ~ 10 in the radiation era [1]. Taking into account this increase of n_l would reinforce our conclusions in making easier (in some parameter range) the detection of GWBs.*(iv)* Finally, we disregarded the possibility of a nonzero cosmological constant which would introduce some quantitative changes in our estimates.

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[13] V. Berezinsky, B. Hnatyk and A. Vilenkin, astro-ph/0001213.

[14] The horizontal lines in Fig. 1 (and Fig. 2) should not be used to estimate the SNR for the detection of the stochastic background, which is optimized by a different filtering technique.

[15] The local maximum of the 1/yr $h^{\rm burst}$ in Fig. 1 corresponds to a redshift $z_m \sim 1$. By contrast, in the model of [13] the (300 times more numerous) GRBs come from a larger volume, up to redshifts ~ 4.

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